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SUPPORTS OF WEIGHTED EQUILIBRIUM MEASURES: COMPLETE CHARACTERIZATION

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ABSTRACT. In this paper, we prove that a compact set $K \subset \mathbb{C}^n$ is the support of a weighted equilibrium measure if and only it is not pluripolar at each of its points extending a result of Saff and Totik to higher dimensions. Thus, we characterize the supports weighted equilibrium measures completely. Our proof is a new proof even in one dimension.

1. Introduction and Background

The supports of weighted extremal measures S_w, are important in pluripotential theory, approximation theory, complex geometry, and they are loosely related to parabolic manifolds [AS11].

Once we know the support of the weighted extremal measure, the weighted extremal function, $V_{K,Q}$, can be determined by solving the homogenous complex Monge-Ampère equation in the bounded components of the complement with boundary value Q. Furthermore, $V_{K,Q} = Q$ on the support S_w quasi everywhere.

Another advantage of determining the supports of weighted extremal measures is as follows: The weighted extremal function of K with respect to Q and the weighted extremal function of the support S_w with respect to the weight $Q|_{S_w}$ are equal. Thus, determining the support of weighted extremal measures makes approximating the weighted capacities very efficient (see [RRR10].)

Some applications in weighted approximation are as follows. By Theorem 2.12 of Appendix B of [ST97], a weighted polynomial attains its essential supremum on the support S_w. In order to make a weighted approximation of a continuous function f on K, f must vanish outside of K. Namely, if f is continuous on K and there is a sequence of weighted polynomials $w^d P_d$ converging uniformly to f on K, then $f \equiv 0$ on $\subset S_w$ (see [ST97, Cal07].)

Since the weighted extremal function $V^*_{K,Q}$ is locally bounded, the weighted extremal measure $(dd^c V^*_{K,Q})^n$ does not put mass on pluripolar sets, i.e., $\operatorname{supp}(dd^c V^*_{K,Q})^n$ is not pluripolar at each of its points; i.e., for all $z \in K$

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and all $r > 0$, $B(z, r) \cap K$ is not pluripolar. It is natural to ask the converse. Namely, if K is a compact set which is not pluripolar at each of its points, then does there exist an admissible weight Q on K such that $\operatorname{supp}(dd^c V^*_{K,Q})^n = K$?

The following theorem gives the converse in $\mathbb{C}$, which characterizes the supports of weighted extremal measures in $\mathbb{C}$.

Theorem 1.1. [ST97, Theorem IV.1.1] *If K is a compact subset of $\mathbb{C}$ which is not pluripolar at each of its points, then there exists an admissible weight on K such that* $\operatorname{supp}(\Delta V_{K,Q}) = K$.

Unfortunately, the proof of the theorem uses logarithmic potentials which is not available in $\mathbb{C}^n$. Branker and the first author investigated the supports of weighted extremal measures (see [Bra04, Ala]. In this paper, we obtain the same theorem 1.1 in $\mathbb{C}^n$as our main result.

First we recall few facts from weighted and unweighted (pluri-)potential theory. Standard references are [Ran95] for unweighted potential theory, [Kli91] for unweighted pluripotential theory, [ST97] for weighted potential theory, and Appendix B in the same book by Thomas Bloom for weighted pluripotential theory.

Let K be a closed subset of $\mathbb{C}^n$. An admissible weight function on K is a lower semicontinuous function $Q : K \to (-\infty, \infty]$ such that

i) $\{z \in K \mid Q(z) < \infty\}$ is not pluripolar.
ii) If K is unbounded, then $Q(z) - \log|z| \to \infty$ as $|z| \to \infty$, $z \in K$.

The function $w = e^{-Q}$ is also used equivalently in the terminology. Especially, the notation w is used more often in weighted approximation (see [Blo09, BL03, ST97].)

The **weighted Siciak-Zahariuta extremal function** of K with respect to Q is defined as

$$V_{K,Q}(z) := \sup\{u(z) \mid u \in L,\ u \le Q \text{ on } K\}. \tag{1.1}$$

Recall that L is the Lelong class:

$$L := \{u \mid u \text{ is plurisubharmonic on } \mathbb{C}^n, u(z) \le \log^+|z| + C_u\}. \tag{1.2}$$

If $Q = 0$, then $V_{K,0}$ is called the **(unweighted) Siciak-Zahariuta extremal function** of K and V_K denotes it.

A compact set K is called **regular** if V_K is continuous. If $K \cap \overline{B(z, r)}$ is regular for all $z \in K$ and $r > 0$, the set K is called **locally regular**. Here we use the notation $B(z_0, r)$ for the open ball of radius r and center z_0.

It is well known that the upper semicontinuous regularization of $V_{K,Q}$ is plurisubharmonic and in L^+ where

$$L^+ := \{u \in L \mid \log^+|z| + C_u \leq u(z)\}.$$

Recall that the **upper semicontinuous regularization** of a function v is defined by $v^*(z) := \limsup_{w \to z} v(w)$.

A subset $P \subset \mathbb{C}^n$ is called **pluripolar** if $E \subset \{z \in \mathbb{C}^n \mid u(z) = -\infty\}$ for some plurisubharmonic function u. If a property holds everywhere except on a pluripolar set we will say that the property holds **quasi everywhere**. It is a well-known fact that $V_{K,Q} = V^*_{K,Q}$ quasi everywhere. See [Kli91].

Let S_w denotes the support of the $(dd^c V^*_{K,Q})^n$, where $(dd^c u)^n$ is the Monge-Ampère measure of u. The following lemma is very useful to determine the supports of Monge-Ampère measures.

Lemma 1.2. [ST97, Appendix B, Theorem 1.3] *Let $S^*_w := \{z \in \mathbb{C}^n \mid V^*_{K,Q}(z) \geq Q(z)\}$. Then we have $S_w \subset S^*_w$.*

Theorem 1.3. [Dem92, Proposition 11.9] *Let u, v be locally bounded plurisubharmonic functions on Ω. Then we have the following inequality*

$$(dd^c \max\{u, v\})^n \geq \chi_{\{u \geq v\}} (dd^c u)^n + \chi_{\{u < v\}} (dd^c v)^n. \tag{1.3}$$

Here χ_A is the **characteristic function** of A. The inequality (1.3) will be called the **Demailly inequality**.

Proposition 1.4. [Sic81, Proposition 2.13] *If K is locally regular and Q is continuous, then $V_{K,Q}$ is continuous.*

2. Characterization of the Supports

Proposition 2.1. *Let K be a non-pluripolar compact set in $\mathbb{C}^n$ and let u be a continuous plurisubharmonic function in Lelong class. If Q is the weight on K defined by $Q := u|_K$, then we have $V_{K,Q} = u$ on K.*

Proof. Because u itself is a competitor in the envelope defining $V_{K,Q}$, we have $u \leq V_{K,Q}$ on $\mathbb{C}^n$; and $V_{K,Q} \leq Q = u$ on K. Thus $V_{K,Q} = u$ on K. □

Note that $u = V^*_{K,Q}$ quasi everywhere on K; i.e., we have $u = V^*_{K,Q}$ on $K \setminus P$ where P is a pluripolar set. The following theorem is our main result which gives the complete characterization of supports of weighted extremal measures.

Theorem 2.2. *Let K be a compact set in $\mathbb{C}^n$ which is not pluripolar at each of its points; i.e., for all $z \in K$ and all $r > 0$, $B(z,r) \cap K$ is not pluripolar. There exists a continuous weight Q on K so that $K = \operatorname{supp}(dd^c V^*_{K,Q})^n$.*

Proof. Since K is compact, $K \subset K_r$ for some $r > 0$, where $K_r := \overline{B(z,r)}$. Let Q_r be the weight on K_r defined by $Q_r := \frac{1}{\sqrt{2r}}|z|^2$. By Example 3.7 of [Ala], we have $\operatorname{supp}(dd^c V_{K_r,Q_r})^n = K_r$.

We define $Q|_K := u = V_{K_r,Q_r}$. By proposition 2.1 we have $V^*_{K,Q} = u$ quasi everywhere on K. By Demailly's inequality we have

$$\begin{aligned}(dd^c V^*_{K,Q})^n &= (dd^c \max\{V^*_{K,Q}, V_{K_r,Q_r})^n \\ &\geq \chi_{\{V_{K_r,Q_r} \geq V^*_{K,Q}\}}(dd^c V_{K_r,Q_r})^n + \chi_{\{V^*_{K,Q} > V_{K_r,Q_r}\}}(dd^c V^*_{K,Q})^n.\end{aligned}$$

Due to the facts that the set $\{V^*_{K,Q} > V_{K_r,Q_r}\} \cap K$ is pluripolar, and that $V^*_{K,Q}$ is locally bounded, we have $(dd^c V^*_{K,Q})^n$ vanishes on $\{V^*_{K,Q} > V_{K_r,Q_r}\} \cap K$. Therefore, we have $(dd^c V^*_{K,Q})^n \geq (dd^c V_{K_r,Q_r})^n$ quasi everywhere on K. Namely, for any non-pluripolar (Borel) subset E of K, we have

$$(dd^c V^*_{K,Q})^n(E) \geq (dd^c V_{K_r,Q_r})^n(E) > 0. \tag{2.1}$$

For any $z \in K$ for every $r > 0$, we have $(dd^c V^*_{K,Q})^n(K \cap B(z,r)) > 0$. Therefore $z \in \operatorname{supp}(dd^c V^*_{K,Q})^n$. □

Corollary 2.3. *Let K be a locally regular compact subset of $\mathbb{C}^n$. Then there exists a continuous weight Q on K such that $K = \operatorname{supp}(dd^c V^*_{K,Q})^n$ and $Q = V_{K,Q}$ on K.*

Proof. We define K_r and Q_r as in the proof of above theorem. By above theorem we have $K = \operatorname{supp}(dd^c V^*_{K,Q})^n$. By Proposition 1.4, we have $V_{K,Q}$ is continuous, thus $V_{K,Q} \leq Q$ on K. By combining these with Lemma 1.2, we have $Q = V_{K,Q}$ on K. □

As a corollary, we obtain the following unexpected result.

Corollary 2.4. *There exists a continuous plurisubharmonic function $u \in L^+$, such that $\operatorname{supp}(dd^c u)^n = \partial\Delta^n$, where Δ^n is the polydisc in $\mathbb{C}^n$.*

Open Problem 2.5. *A compact set $K \subset \mathbb{C}^n$ is locally regular if and only it is the support of the Monge-Ampère measure of a continuous function in L^+.*

Remark 2.6. Note that the above open problem might be a step to understand the measures which are Monge-Ampère measures of continuous plurisubharmonic function.

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